

A Pseudoclassical Model for P, T –Invariant Planar Fermions

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Abstract

A pseudoclassical model is proposed for the description of planar P, T –invariant massive fermions. The quantization of the model leads to the (2+1)-dimensional P, T –invariant fermion model used recently in P, T –conserving theories of high- T_c superconductors. The rich symmetry of the quantum model is elucidated through the analysis of the canonical structure of its pseudoclassical counterpart. We show that both the quantum P, T –invariant planar massive fermion model and the proposed pseudoclassical model — for a particular choice of the parameter appearing in the Lagrangian — have a $U(1,1)$ dynamical symmetry as well as an $N = 3$ supersymmetry. The hidden supersymmetry leads to a non-standard superextension of the (2+1)-dimensional Poincaré group. In the quantum theory the one particle states provide an irreducible representation of the extended supergroup labelled by the zero eigenvalue of the superspin. We discuss the gauge modification of the pseudoclassical model and compare our results with those obtained from the standard pseudoclassical model for massive planar fermions.

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1 Introduction

Planar gauge field theories have many interesting features [1]. Especially in their use for constructing models of high- T_c superconductors [2] one is interested in having a P - and T -invariant system of topologically massive gauge fields and a P - and T -invariant system of massive Dirac spinor fields. For this purpose, one usually introduces doublets of these fields with mass terms having opposite sign [2, 3]. In a recent paper [4] the hidden dynamical symmetries of the simplest P - and T -invariant free fermion theory were investigated. The model [4] is described by the Lagrangian

$$\mathcal{L} = \bar{\psi}_u(p\gamma + m)\psi_u + \bar{\psi}_d(p\gamma - m)\psi_d \quad (1.1)$$

and it is invariant under a global $U_c(1)$ symmetry describing chiral rotations to which it is associated the conserved chiral current $I_\mu = \frac{1}{2}(\bar{\psi}_u\gamma_\mu\psi_u - \bar{\psi}_d\gamma_\mu\psi_d)$. A very interesting feature of the model described by (1.1) is the appearance of a hidden $N = 3$ supersymmetry which yields a non-standard superextension of the Poincaré group so that the one particle states associated to (1.1) realize an irreducible representation of the Poincaré supergroup, labelled by the zero eigenvalue of the superspin operator [4].

In a beautiful paper [5] Gibbons, Rietijk and van Holten investigated space-time symmetries in terms of the motion of pseudoclassical spinning point particles [6, 7, 8]. In their analysis they revealed the existence of a non-standard supersymmetry characterized by the fact that the Poisson brackets of the odd Grassmann generators yield an even integral of motion different from the Hamiltonian of the system. A non-standard supersymmetry of the same kind is useful to describe the hidden symmetries [9] of a 3-dimensional monopole [10].

In this paper we shall show that the hidden $N = 3$ supersymmetry of the P, T -invariant 3D fermion model described by (1.1) may be understood in a natural way starting with the pseudoclassical model of a relativistic spinning particle proposed in ref. [11].

The pseudoclassical model of ref. [11] is defined by the Lagrangian

$$L = \frac{1}{2e}(\dot{x}_\mu - iv\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda)^2 - \frac{e}{2}m^2 - 2\nu m v\theta^+\theta^- - \frac{i}{2}\xi^\mu\dot{\xi}_\mu + \frac{i}{2}\theta_a\dot{\theta}_a. \quad (1.2)$$

A complete description of the quantities appearing in (1.2) is given in section 2. Here we only notice that the pseudoclassical model (1.2) depends on a parameter ν . As we shall see, there is a special value of the parameter ν ($\nu = 1$) which leads to a maximal symmetry of the classical system.

Since the pseudoclassical model (1.2) has an even nilpotent constraint, it is similar to the class of pseudoclassical models used in describing particles with spin $s \geq 1$ [12]. Its main difference with the standard pseudoclassical model (SPM) for the massive Dirac particle [7] lies in the fact that the odd constraint of the latter model is replaced in [11] by the even nilpotent constraint. We shall show that, although at the quantum level both models lead to the P, T -invariant planar model described by (1.1), only the model of ref. [11] reproduces classically the same symmetries of its quantum counterpart. For the 3D standard pseudoclassical model [7] the situation is quite reminiscent of what occurs in systems with a quantum anomaly, since the classical symmetries are not preserved by the quantization procedure. Unlike the SPM [7], the model considered in this paper naturally yields a $U(1)$ gauge theory with the gauge field coupled to the $U_c(1)$ chiral current and it allows for a

nontrivial interplay between local and discrete symmetries on one hand, and ordinary and graded symmetries on the other hand.

The paper is organized as follows. In order to have a self-contained presentation, we introduce in section 2 the pseudoclassical model of ref. [11]. In this section we analyze its global, local and discrete symmetries within the framework of the Lagrangian formalism. Section 3 is devoted to the Hamiltonian description of the system. First we get the equations of motion and find the integrals of motion which are the generators of the continuous symmetries described in section 2. Then, we notice that for a special value of the parameter of the pseudoclassical model ($\nu = 1$) the system has two additional (local in the evolution parameter) integrals of motion. We find that the integrals of motion of the system form a broader $U(1,1)$ symmetry and a hidden $N = 3$ supersymmetry, and present both symmetries in a Poincaré covariant form. Section 4 is devoted to gauging the $U(1)$ symmetry corresponding at the quantum level to the chiral rotations. In section 5 we quantize the model. We show that $\nu = 1$ is also special from the point of view of the quantum theory: namely, we demonstrate that only for $\nu = 1$ the quantization procedure preserves the P - and T - symmetries of the classical model. For this value of the parameter ν the $U(1,1)$ symmetry and the $N = 3$ supersymmetry are realized in the quantum theory and the $N = 3$ supersymmetry leads to the non-standard superextension of the Poincaré group. Finally, we investigate the quantum theory of the model with chiral $U(1)$ gauge symmetry. Section 6 is devoted to some concluding remarks.

In Appendix A we evidence a hidden symmetry of the pseudoclassical model existing in the subspace of the Grassmann (pseudo)scalar variables.

The analysis of the planar SPM [7] and its dynamical symmetries is presented in Appendix B.

2 The model and its Lagrangian symmetries

2.1 The model

The planar model analyzed in ref. [11] is defined by the action

$$A = \int_{\tau_i}^{\tau_f} L d\tau + B, \quad (2.1)$$

where $L = L_0 + L_\xi + L_\theta$ is the Lagrangian,

$$L_0 = \frac{1}{2e}(\dot{x}_\mu - iv\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda)^2 - \frac{e}{2}m^2 - 2\nu mv\mathcal{N}, \quad (2.2)$$

$$\mathcal{N} = \theta^+\theta^- = -i\theta_1\theta_2, \quad (2.3)$$

$$L_\xi = -\frac{i}{2}\xi_\mu\dot{\xi}^\mu, \quad L_\theta = \frac{i}{2}\theta_a\dot{\theta}_a, \quad (2.4)$$

and $B = B_\xi + B_\theta$ is the boundary term,

$$B_\xi = -\frac{i}{2}\xi_\mu(\tau_f)\xi^\mu(\tau_i), \quad B_\theta = \frac{i}{2}\theta_a(\tau_f)\theta_a(\tau_i). \quad (2.5)$$

In (2.2) m is a mass parameter and $\nu \neq 0$ is a dimensionless parameter. We use the metric $\eta_{\mu\nu} = \text{diag}(-, +, +)$ and the totally antisymmetric tensor $\epsilon_{\mu\nu\lambda}$, $\epsilon^{012} = 1$.

The configuration superspace of the model is described by the set of variables x_μ , ξ_μ , θ^\pm , e and v . We denote with x_μ the space-time coordinates of the particle; together with the scalar Lagrange multipliers e and v they are real even variables. ξ_μ is a real odd (Grassmann) vector whereas θ^+ and θ^- are mutually conjugate odd scalar variables related to the real variables θ_a , $a = 1, 2$, by the equation

$$\theta^\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2). \quad (2.6)$$

We shall specify later the transformation properties of all the variables under P and T inversions.

The form of the kinetic terms L_ξ and L_θ (as well as that of the boundary terms B_ξ and B_θ) manifests the fact that the scalar variables θ_a , $a = 1, 2$, are “timelike” in contrast to the one “spacelike” scalar variable ξ_* used in the standard pseudoclassical formulation of the massive spin-1/2 particle (see Appendix B).

The inclusion of the boundary terms in the action (2.1) is needed since the equations for the Grassmann variables are first order [13]. As a result, the classical solutions to these equations extremize the action (2.1) with the boundary conditions:

$$\delta\eta_A(\tau_f) + \delta\eta_A(\tau_i) = 0, \quad \eta_A = \xi_\mu, \theta_a. \quad (2.7)$$

The change of variables $\theta_1, \theta_2 \rightarrow \theta_1, -\theta_2$, and, correspondingly, $\theta^\pm \rightarrow \theta^\mp$, amounts only to changing $\mathcal{N} \rightarrow -\mathcal{N}$ in the action (2.1). Without loss of generality, one may then always choose

$$\nu > 0, \quad (2.8)$$

since the model with $\nu < 0$ is reproduced by the above change of the variables.

2.2 Global symmetries

The action (2.1) is Poincaré-invariant. In addition, it is invariant under the following global super-transformations:

$$\delta\xi_\mu = \epsilon_\mu, \quad \delta x_\mu = 2i\epsilon_{\mu\nu\lambda}\epsilon^\nu \cdot \int_{\tau_i}^\tau v(\tau')\xi^\lambda(\tau')d\tau', \quad (2.9)$$

where ϵ_μ is a constant odd vector infinitesimal transformation parameter. Unlike the ordinary global super-transformations, the transformations (2.9) have a nonlocal character in the evolution parameter. As a consequence,

$$(\delta_1\delta_2 - \delta_2\delta_1)x_\mu = -4i\epsilon_{\mu\nu\lambda}\epsilon_1^\nu\epsilon_2^\lambda \cdot \int_{\tau_i}^\tau v(\tau')d\tau' \neq \text{const.}$$

Thus, the commutator of two global super-transformations (2.9) does not yield the space-time translations characterized by a constant nilpotent vector as it happens for the standard global super-transformation in systems with a supersymmetric spectrum.

The action is invariant under another set of global super-transformations

$$\delta_\epsilon x_\mu = i\epsilon \xi_\mu, \quad \delta_\epsilon \xi_\mu = \epsilon e^{-1}(\dot{x}_\mu - iv\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda) \quad (2.10)$$

with a constant scalar odd parameter ϵ . In the standard pseudoclassical model of a spin-1/2 particle [7], there is *local* supersymmetry (with a local infinitesimal parameter $\epsilon = \epsilon(\tau)$) induced by transformations similar in form to eq. (2.10).

The action is also invariant under

$$\delta_\epsilon \xi_\mu = i\epsilon \epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda \quad (2.11)$$

with the odd scalar infinitesimal transformation parameter ϵ .

Furthermore, there is invariance under the global U(1) transformations

$$\delta_\gamma \theta^\pm = \pm i\gamma \theta^\pm \quad (2.12)$$

(or corresponding SO(2) transformations in terms of the real variables θ_a), and the global even transformations

$$\delta_\omega x_\mu = i\omega \epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda, \quad \delta_\omega \xi_\mu = 2\omega e^{-1}\epsilon_{\mu\nu\lambda}\dot{x}^\nu\xi^\lambda, \quad (2.13)$$

characterized by the even scalar transformation parameters γ and ω , respectively. In Section 3 we shall analyse these global symmetries within the Hamiltonian formalism.

2.3 Local symmetries

There are two local symmetries of the action. They are given by the transformation

$$\begin{aligned} \delta_\alpha E &= \frac{d}{d\tau}(\alpha E), & E &= e, v, \\ \delta_\alpha X &= \alpha \dot{X}, & X &= x_\mu, \xi_\mu, \theta_a, \end{aligned} \quad (2.14)$$

which is the reparametrization transformation, and by the transformation

$$\begin{aligned} \delta_\beta e &= 0, & \delta_\beta v &= \dot{\beta}, & \delta_\beta \theta^\pm &= \pm im\beta \theta^\pm, \\ \delta_\beta x_\mu &= i\beta \epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda, & \delta_\beta \xi_\mu &= 2\beta e^{-1}\epsilon_{\mu\nu\lambda}\dot{x}^\nu\xi^\lambda, \end{aligned} \quad (2.15)$$

to which we shall refer to as the ‘ β -transformation’. In eqs. (2.14), (2.15) α and β are the infinitesimal transformation parameters. Since we have $\delta_\alpha L = (d/d\tau)(\alpha L)$ and $\delta_\beta L = (d/d\tau)(i\beta e^{-1}\epsilon_{\mu\nu\lambda}\dot{x}^\mu\xi^\nu\xi^\lambda)$, the action (2.1) is invariant under (2.14) and (2.15) if one requires

$$\alpha(\tau_i) = \alpha(\tau_f) = 0, \quad \beta(\tau_i) = \beta(\tau_f) = 0. \quad (2.16)$$

Formally ($\beta \neq const$), the β -transformation is the linear combination of the transformations (2.12) and (2.13) with $m^{-1}\gamma = \omega$. The β -transformation mixes the even coordinates x_μ with the odd spin variables ξ_μ ; this transformation is characterized by an even parameter and, as we shall see in Section 3, it is generated by an even (nilpotent) constraint. As a consequence, one has $\delta_{\beta_1}\delta_{\beta_2} - \delta_{\beta_2}\delta_{\beta_1} = 0$.

2.4 Discrete symmetries

The action is invariant under the discrete P - and T -transformations:

$$P : X_\mu \rightarrow (X_0, -X_1, X_2), \quad T : X_\mu \rightarrow (-X_0, X_1, X_2), \quad (2.17)$$

where $X_\mu = x_\mu, \xi_\mu$, and

$$P : , T : (e, v) \rightarrow (e, -v), \quad (2.18)$$

$$P : \theta_a \rightarrow (\theta_1, -\theta_2). \quad T : \theta_a \rightarrow (-\theta_1, \theta_2). \quad (2.19)$$

Since $P : \theta^\pm \rightarrow \theta^\mp$, and $T : \theta^\pm \rightarrow -\theta^\mp$, one has that $P : , T : \mathcal{N} \rightarrow -\mathcal{N}$.

Due to the global $U(1)$ symmetry, there is a freedom in the choice of the form of the P and T transformations in the sector of the variables θ_a .

If one considers only the continuous and discrete symmetries described so far, the parameter ν can take any value in the classical theory. As we shall see in Appendix A, there is a hidden classical symmetry restricting the allowed values of ν . This symmetry, realized in the subspace of the variables θ^\pm , is the product of P or T (2.19) with a special $U(1)$ transformation.

3 Hamiltonian description

3.1 Equations and integrals of motion

Let us turn to the canonical description of the model and to the construction of the general solution to the classical equations of motion.

The nontrivial brackets of the system are the following:

$$\begin{aligned} \{x_\mu, p_\nu\} &= \eta_{\mu\nu}, \quad \{e, p_e\} = 1, \quad \{v, p_v\} = 1, \\ \{\xi_\mu, \xi_\nu\} &= i\eta_{\mu\nu}, \quad \{\theta_a, \theta_b\} = -i\delta_{ab}, \end{aligned} \quad (3.1)$$

and, hence,

$$\{\theta^+, \theta^-\} = -i. \quad (3.2)$$

In correspondence with the two local symmetries of the Lagrangian, in the Hamiltonian formalism one has two secondary first class constraints

$$\phi = \frac{1}{2}(p^2 + m^2) \approx 0, \quad (3.3)$$

$$\chi = i\epsilon_{\mu\nu\lambda} p^\mu \xi^\nu \xi^\lambda + 2\nu m \mathcal{N} \approx 0, \quad (3.4)$$

which are the generators of the transformations (2.14), (2.15) and form the trivial algebra

$$\{\phi, \chi\} = 0. \quad (3.5)$$

In addition, there are two primary first class constraints $p_e \approx 0$ and $p_v \approx 0$ merely stating the fact that e and v are Lagrange multipliers. The Hamiltonian is a linear combination of the constraints [14]:

$$H = e\phi + v\chi + u_1 p_e + u_2 p_v, \quad (3.6)$$

with $u_{1,2} = u_{1,2}(\tau)$ arbitrary functions of the evolution parameter τ .

The equations of motion have the form:

$$\begin{aligned}\dot{x}_\mu &= ep_\mu + iv\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda, & \dot{p}_\mu &= 0, \\ \dot{\xi}_\mu &= 2v\epsilon_{\mu\nu\lambda}p^\nu\xi^\lambda, & \dot{\theta}^\pm &= \pm 2ivm\nu\theta^\pm,\end{aligned}\tag{3.7}$$

and $\dot{e} = u_1$, $\dot{v} = u_2$. From (3.7), one identifies the set of integrals of motion :

$$P_\mu = p_\mu, \quad J_\mu = -\epsilon_{\mu\nu\lambda}x^\nu p^\lambda - \frac{i}{2}\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda,\tag{3.8}$$

$$\Gamma = p\xi, \quad \Xi = i\epsilon_{\mu\nu\lambda}\xi^\mu\xi^\nu\xi^\lambda, \quad \mathcal{N} = \theta^+\theta^-, \quad \Delta = i\epsilon_{\mu\nu\lambda}p^\mu\xi^\nu\xi^\lambda.\tag{3.9}$$

The integrals P_μ and J_μ are the generators of the Poincaré transformations, whereas the integrals (3.9) generate the global symmetries given by the transformations (2.10)–(2.13). The two last integrals of motion, \mathcal{N} and Δ , are related by the constraint (3.4) which is the generator of the local β –transformations. This explains why the global symmetries (2.12) and (2.13) and the local symmetry (2.15) are related.

Using the mass shell constraint, one may introduce the complete oriented triad $e_\mu^{(\alpha)} = e_\mu^{(\alpha)}(p)$, $\alpha = 0, 1, 2$,

$$e_\mu^{(0)} = \frac{p_\mu}{\sqrt{-p^2}}, \quad e_\mu^{(\alpha)}\eta_{\alpha\beta}e_\nu^{(\beta)} = \eta_{\mu\nu}, \quad \epsilon_{\mu\nu\lambda}e^{(0)\mu}e^{(i)\nu}e^{(j)\lambda} = \epsilon^{0ij}.\tag{3.10}$$

Upon defining $\xi^{(\alpha)} = \xi^\mu e_\mu^{(\alpha)}$ and

$$\xi^{(\pm)} = \frac{1}{\sqrt{2}}(\xi^{(1)} \pm i\xi^{(2)}),\tag{3.11}$$

one has

$$\{\xi^{(+)}, \xi^{(-)}\} = i,\tag{3.12}$$

which differs in sign from the brackets (3.2) for the variables θ^\pm . This difference is relevant for the symmetries of the model.

The space-like components of the triad $e_\mu^{(i)}$, $i = 1, 2$, are not Lorentz vectors (see, e.g. ref. [15]), and, therefore, the quantities $\xi^{(i)}$ are not scalars.

Taking into account the mass shell constraint, one can present the nilpotent constraint (3.4) in the equivalent form:

$$\mathcal{X} = \xi^{(+)}\xi^{(-)} - \nu\theta^+\theta^- \approx 0.\tag{3.13}$$

With the help of the triad (3.10) one finds yet another dependence among the integrals of motion (3.8), (3.9). Taking into account the completeness of the triad, one gets the equality:

$$\Xi = 3i(P^2)^{-1}\Gamma\Delta.\tag{3.14}$$

As a result, one has the following combination of the integrals of motion:

$$\rho = \nu \cdot p\xi\theta^+\theta^- - im\kappa \cdot \epsilon_{\mu\nu\lambda}\xi^\mu\xi^\nu\xi^\lambda,\tag{3.15}$$

which is an odd function and has the weakly vanishing bracket

$$\{\rho, \rho\} = -\frac{3}{2}i\kappa\chi^2.$$

Using (3.14), on the surface of the mass shell constraint, for $\kappa = 1/6$, one finds

$$\rho = -m\xi^{(0)} \cdot (\xi^{(+)}\xi^{(-)} - \nu\theta^+\theta^-). \quad (3.16)$$

Thus, for $\kappa = 1/6$, ρ is proportional to χ and the additional constraint

$$\rho \approx 0 \quad (3.17)$$

can be imposed.

The classical odd constraint (3.17) has the solution $\xi^{(0)} = 0$ in addition to the subspace singled out by the nilpotent constraint (3.4); consequently, its introduction modifies the physical content of the original model. At the quantum level, the analog of $\xi^{(0)}$ is an invertible operator; this implies that the quantum analog of the constraint (3.16) is equivalent to the quantum analog of the constraint (3.4). In the quantum theory, the additional constraint (3.17) does not change the physical content of the model. Of course, one may exclude the point $\xi^{(0)} = 0$ from the phase space of the classical system requiring that $\xi^{(0)} \neq 0$; in this case, the additional constraint $\rho \approx 0$, for $\kappa = 1/6$, does not change the physical content of the model even at the classical level. From this one learns that the physical content of the modified classical model — i.e. the model supplied with the constraint (3.15), (3.17) — is governed by the c -number parameter κ and that, for the particular value $\kappa = 1/6$, the additional odd constraint differs from the even nilpotent constraint (3.4) only in the factor $\xi^{(0)}$. Hence, one may change Grassmann parity of the integrals of motion multiplying them by the odd integral $\xi^{(0)}$. These observations are helpful for unveiling the hidden continuous symmetries of the model.

3.2 Solution to the equations and additional integrals of motion

Let us go back to the discussion of our original model defined by the constraints (3.3), (3.4) and by the Hamiltonian (3.6).

Since $\xi^{(0)} = \text{const}$, the equations of motion for the odd variables

$$\xi_\mu = -\xi^{(0)}e_\mu^{(0)} + \xi^{(i)}e_\mu^{(i)} \quad (3.18)$$

have the same form (up to the constant ν) as the equations for the variables θ^\pm . Namely,

$$\dot{\xi}^{(\pm)} = \pm 2imv\xi^{(\pm)}. \quad (3.19)$$

From (3.19) and (3.7) one concludes that, if

$$\nu = 1, \quad (3.20)$$

the odd variables θ^\pm and $\xi^{(\pm)}$ have a harmonic-like evolution law with equal (in general time-dependent) angular velocities. As a result, for $\nu = 1$ one has two additional mutually conjugate integrals of motion:

$$\mathcal{V}_+ = i\xi^{(+)}\theta^-, \quad \mathcal{V}_- = \mathcal{V}_+^* = i\xi^{(-)}\theta^+, \quad (3.21)$$

which generate the global symmetries of the model.

Except when explicitly stated otherwise, in the following we shall fix $\nu = 1$. For this value of the parameter the model is maximally symmetric, i.e. it has the maximal number of integrals of motion.

The general solution to the equations of motion is given by

$$\xi^{(\pm)}(\tau) = \xi^{(\pm)}(\tau_i) e^{\pm i\Omega(\tau; \tau_i)}, \quad \theta^\pm(\tau) = \theta^\pm(\tau_i) e^{\pm i\Omega(\tau; \tau_i)}, \quad (3.22)$$

$$x_\mu(\tau) = p_\mu \cdot \int_{\tau_i}^{\tau} e(\tau') d\tau' + e_\mu^{(i)} \left(x_Z^{(i)}(\tau) - x_Z^{(i)}(\tau_i) \right) + x_\mu(\tau_i), \quad (3.23)$$

$$x_Z^{(i)}(\tau) = -im^{-1} \xi^{(0)} \xi^{(i)}(\tau). \quad (3.24)$$

$\Omega(\tau; \tau_i)$ is defined by the relation

$$\Omega(\tau; \tau_i) = 2m \int_{\tau_i}^{\tau} v(\tau') d\tau'. \quad (3.25)$$

We assume that the constraints (3.3), (3.4) are satisfied by p_μ and by the initial values of the variables θ^\pm and $\xi^{(\pm)}$, i.e. by $\theta^\pm(\tau_i)$ and $\xi^{(\pm)}(\tau_i)$ [14]. In writing the solution to the equations of motion for the ‘oscillating’ odd variables θ^\pm and $\xi^{(\pm)}$, the pertinent boundary conditions (2.7) have been taken into account. This point is discussed in detail in the Appendix A.

Eq. (3.24) describes the classical analog of the quantum Zitterbewegung [15]. From eqs. (3.23) and (3.24) one finds that

$$Z^i = x^{(i)} + im^{-1} \xi^{(0)} \xi^{(i)}$$

are integrals of motion.

The evolution of the odd vector variable ξ_μ is written in covariant form as $\xi_\mu(\tau) = G_{\mu\nu}(\tau, \tau_i) \xi^\nu(\tau_i)$, with $G_{\mu\nu}(\tau, \tau_i)$ given by

$$G_{\mu,\nu}(\tau, \tau_i) = -e_\mu^{(0)} e_\nu^{(0)} + \left(\eta_{\mu\nu} + e_\mu^{(0)} e_\nu^{(0)} \right) \cos \Omega(\tau; \tau_i) + \epsilon_{\mu\nu\lambda} e^{(0)\lambda} \sin \Omega(\tau; \tau_i).$$

When $\nu \neq 1$ one may construct the integrals of motion:

$$\mathcal{V}_{\nu+} \equiv i\xi^{(+)}\theta^- \exp(i(\nu-1)\Omega(\tau; \tau_i)), \quad \mathcal{V}_{\nu-} = \mathcal{V}_{\nu+}^*,$$

which are *nonlocal (in τ) functions* due to the presence of the factor $\exp(i(\nu-1)\Omega(\tau; \tau_i))$. They become local integrals (3.21) when $\nu = 1$.

3.3 Global U(1,1) symmetry and $N = 3$ supersymmetry

Let us consider the following linear combinations of the nilpotent even integrals of motion \mathcal{V}_+ , \mathcal{V}_- , Δ and \mathcal{N} :

$$\begin{aligned} R_0 &= -\frac{1}{2}(\xi^{(+)}\xi^{(-)} + \theta^+\theta^-), \\ R_1 &= \frac{i}{2}(\xi^{(+)}\theta^- + \xi^{(-)}\theta^+), \quad R_2 = \frac{1}{2}(\xi^{(+)}\theta^- - \xi^{(-)}\theta^+). \end{aligned} \quad (3.26)$$

They are real quantities, $R_\alpha^* = R_\alpha$, and satisfy the $su(1,1)$ algebra

$$\{R_\alpha, R_\beta\} = -\epsilon_{\alpha\beta\gamma} R^\gamma. \quad (3.27)$$

Moreover, they are in involution with the nilpotent constraint

$$\mathcal{X} = \xi^{(+)}\xi^{(-)} - \theta^+\theta^- \approx 0, \quad (3.28)$$

since $\{R_\alpha, \mathcal{X}\} = 0$. Therefore, the set of integrals of motion R_α and \mathcal{X} form a $u(1,1) = su(1,1) \times u(1)$ algebra. The algebra of the pseudounitary group $U(1,1)$ appears since the pairs of variables θ^\pm and $\xi^{(\pm)}$ have brackets differing in sign. Due to eq. (3.7), \mathcal{X} plays the role of the Hamiltonian for the odd variables ξ_μ and θ^\pm .

The Casimir operator of the group $SU(1,1)$, $C = R_\alpha R^\alpha$,

$$\{C, R_\alpha\} = 0, \quad (3.29)$$

takes the value

$$C = -\frac{3}{2}\xi^{(+)}\xi^{(-)}\theta^+\theta^- \approx 0,$$

and it is related to R_α through the classical equation:

$$R_\alpha R_\beta = \eta_{\alpha\beta} \cdot \frac{1}{3}C. \quad (3.30)$$

The integral R_0 can be written in the manifestly covariant form

$$R_0 = \frac{1}{2} \left(\frac{i}{2\sqrt{-p^2}} \epsilon_{\nu\rho\lambda} p^\nu \xi^\rho \xi^\lambda - \theta^+ \theta^- \right),$$

and the integrals $R_{1,2}$ can be obtained from the Lorentz vector

$$R_\mu^\perp = \frac{i}{2} \left(\xi_\mu - p_\mu \frac{p\xi}{p^2} \right) \theta_1 - \frac{i}{2\sqrt{-p^2}} \epsilon_{\mu\nu\lambda} p^\nu \xi^\lambda \theta_2, \quad (3.31)$$

which satisfies the relations $R_\mu^\perp p^\mu = 0$ and $R_\mu^\perp e^{(i)\mu} = R_i$. One can now construct the vector

$$\mathcal{R}_\mu = R_\mu^\perp + e_\mu^{(0)} R_0, \quad (3.32)$$

providing the covariant set of $SU(1,1)$ generators, which, due to $\mathcal{R}^\mu e_\mu^{(\alpha)} = R^\alpha$, is equivalent to the noncovariant set (3.26).

Let us consider the following odd integrals of motion, which are the composition of the integrals (3.32) and $\xi^{(0)}$:

$$\Theta_\mu = \xi^{(0)} \mathcal{R}_\mu. \quad (3.33)$$

Due to (3.30), the odd integrals (3.33) together with the even integral C satisfy the covariant algebra of an $N = 3$ supersymmetry:

$$\{\Theta_\mu, \Theta_\nu\} = -\eta_{\mu\nu} \cdot \frac{i}{3}C, \quad \{\Theta_\mu, C\} = 0. \quad (3.34)$$

The relation between the even integrals, \mathcal{R}_μ , and the odd ones, Θ_μ , is similar to the one existing between the constraint χ and the additional constraint ρ .

In this section we have unveiled the global $U(1,1)$ symmetry and the $N = 3$ supersymmetry of the classical model. The even generator of the supersymmetry, C , differs from the constraint \mathcal{X} which is, in fact, a Hamiltonian for all the Grassmann variables of the system. This non-standard supersymmetry is analogous to that discovered in ref. [5].

3.4 Covariant form of the U(1,1) symmetry

With the help of the standard prescription [14] one finds the form of the global transformations generated by the even, \mathcal{R}_μ , and odd, Θ_μ , integrals of motion:

$$\delta_{\omega_\mu} X = \omega_\mu \{X, \mathcal{R}^\mu\}, \quad \delta_{\epsilon_\mu} X = \epsilon_\mu \{X, \Theta^\mu\}, \quad X = x_\mu, \xi_\mu, \theta_a. \quad (3.35)$$

In (3.35) ω_μ and ϵ_μ are even and odd constant infinitesimal transformation parameters, respectively. Both transformations (3.35) mix the even coordinates x_μ with the ‘internal’ translation-invariant odd variables of the model. Since the explicit form of the transformation properties of the coordinates x_μ is not needed, we shall not write it here.

A more detailed analysis of the transformations (3.35) for the odd variables gives the possibility to ‘covariantize’ the U(1,1) symmetry of the theory. For this purpose, one introduces the following linear combinations Σ_a^i , $a = 1, 2$, $i = 1, 2$, of the odd variables $\xi^{(i)}$ and θ_a ,

$$\begin{aligned} \Sigma_1^1 &= \xi^{(1)} + \theta_1, & \Sigma_2^1 &= -\xi^{(2)} + \theta_2, \\ \Sigma_1^2 &= \xi^{(2)} + \theta_2, & \Sigma_2^2 &= \xi^{(1)} - \theta_1. \end{aligned} \quad (3.36)$$

These variables are spinors (in the index a) with respect to the action of the SU(1,1) generators R_α and form SO(2) vectors (in the index i) with respect to the action of the generator \mathcal{X} . This gives the possibility to ‘covariantize’ the SU(1,1) generators R_α as well as the SO(2) (or U(1)) generator \mathcal{X} . In fact, let us introduce the SO(2) tensor $\epsilon^{ij} = -\epsilon^{ji}$ and the SU(1,1) spinor tensor $\tilde{\epsilon}^{ab} = -\tilde{\epsilon}^{ba}$, as well as its inverse, $\tilde{\epsilon}_{ab} = -\tilde{\epsilon}_{ba}$, normalized as $\epsilon^{12} = \tilde{\epsilon}^{12} = \tilde{\epsilon}_{12} = 1$, and let us define $\Sigma^{ia} = \tilde{\epsilon}^{ab} \Sigma_b^i$. Lowering and raising the ‘spinor’ indices is understood only for the variables Σ_a^i , but not for $\xi^{(1,2)}$ and $\theta_{1,2}$. The new variables have the following brackets:

$$\{\Sigma^{ia}, \Sigma_b^j\} = -2i\epsilon^{ij}\delta_b^a. \quad (3.37)$$

In terms of Σ_a^i the U(1,1) generators are written in the ‘covariant’ form,

$$R_\alpha = -\frac{1}{8}\epsilon^{ij}\Sigma^{ia}(\tilde{\gamma}_\alpha)_a{}^b\Sigma_b^j, \quad (3.38)$$

$$\mathcal{X} = -\frac{i}{4}\Sigma^{ia}\Sigma_a^i. \quad (3.39)$$

In eqs. (3.38) we have introduced the γ -matrices in the Majorana representation, $(\tilde{\gamma}^0)_a{}^b = -(\sigma^2)_a{}^b$, $(\tilde{\gamma}^1)_a{}^b = i(\sigma^3)_a{}^b$, $(\tilde{\gamma}^2)_a{}^b = -i(\sigma^1)_a{}^b$, $\tilde{\gamma}_\alpha\tilde{\gamma}_\beta = -\eta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\tilde{\gamma}^\gamma$, and we used the tilde to stress that they are constant in the tangent space and differ from the matrices $\gamma^{(\alpha)} = e_\mu^{(\alpha)}\gamma^\mu$. The variables Σ_a^i are indeed SU(1,1) spinors and SO(2) vectors:

$$\{\Sigma_b^i, R_\alpha\} = \frac{i}{2}(\tilde{\gamma}_\alpha)_a{}^b\Sigma_b^i, \quad (3.40)$$

$$\{\Sigma_a^i, \mathcal{X}\} = \epsilon^{ij}\Sigma_a^j. \quad (3.41)$$

Defining the even variables

$$\Xi_a^i = i\xi^{(0)}\Sigma_a^i, \quad (3.42)$$

one readily finds that they satisfy to

$$\{\Xi_b^i, R_\alpha\} = \frac{i}{2}(\tilde{\gamma}_\alpha)_a{}^b \Xi_b^i, \quad (3.43)$$

$$\{\Xi_a^i, \mathcal{X}\} = \epsilon^{ij} \Xi_a^j. \quad (3.44)$$

Taking into account that

$$i\{\Xi_a^i, \Xi_b^j\} = \Sigma_a^i \Sigma_b^j,$$

one gets that (3.38) and (3.39) can be written as:

$$R_\alpha = -\frac{i}{8}\epsilon^{ij}(\tilde{\gamma}_\alpha)_a{}^b \{\Xi^{ia}, \Xi_b^j\}, \quad (3.45)$$

$$\mathcal{X} = \frac{1}{4}\{\Xi^{ia}, \Xi_a^i\}. \quad (3.46)$$

Eqs. (3.43)–(3.46) express covariantly the U(1,1) symmetry of the system.

4 Gauging the U(1) symmetry

The action is invariant under the global U(1) transformations (2.12) generated by the integral of motion \mathcal{N} :

$$\theta^\pm \rightarrow \theta^{\pm'} = e^{\pm i\gamma} \theta^\pm. \quad (4.1)$$

Let us assume that γ is not a constant, i.e. $\dot{\gamma} \neq 0$. Since under (4.1) one has $\delta_\gamma L_\theta = d\gamma/d\tau \cdot \mathcal{N}$, the lagrangian of the system is not invariant. In addition, if one assumes that $\gamma = \gamma(x_\mu(\tau))$, one has $\delta_\gamma L = \partial_\mu \gamma \cdot \dot{x}^\mu \mathcal{N}$. Upon introducing a ‘compensating’ term

$$L_{int}^1 = \mathcal{Q} \mathcal{N} \dot{x}^\mu \mathcal{A}_\mu \quad (4.2)$$

in the Lagrangian, with \mathcal{Q} a coupling constant and $\mathcal{A}_\mu = \mathcal{A}_\mu(x)$ a gauge field transforming as

$$\delta_\gamma \mathcal{A}_\mu = -\mathcal{Q}^{-1} \partial_\mu \gamma(x), \quad (4.3)$$

one easily verifies that $\delta_\gamma(L + L_{int}^1) = 0$. The term L_{int}^1 does not change the reparametrization invariance, but it violates the β –invariance of the action. To restore the latter invariance, one introduces the additional term

$$L_{int}^2 = \frac{e}{2} \mathcal{Q} \mathcal{N} i \xi_\mu \xi_\nu \mathcal{F}^{\mu\nu}, \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \quad (4.4)$$

which is invariant under the U(1) gauge transformations (4.1), (4.3). Since the Lagrangian

$$L_{int} = L_{int}^1 + L_{int}^2 = \mathcal{Q} \mathcal{N} \left(\dot{x}_\mu \mathcal{A}^\mu + \frac{e}{2} i \xi_\mu \xi_\nu \mathcal{F}^{\mu\nu} \right) \quad (4.5)$$

satisfies the condition that $\delta_\beta L_{int} = \mathcal{Q} \frac{d}{d\tau} (\mathcal{N} \delta_\beta x_\mu \mathcal{A}^\mu)$, $L + L_{int}$ is invariant under the local U(1) and β – symmetries.

The boundary term B_θ is invariant under the local U(1) transformations if the function $\gamma(x)$ satisfies the boundary condition

$$\Delta\gamma = \gamma(x(\tau_f)) - \gamma(x(\tau_i)) = 2\pi n, \quad n \in Z. \quad (4.6)$$

The U(1) gauge interaction preserves the invariance of the action under the discrete P and T inversions if one supplements the transformation laws (2.17)–(2.19) with

$$P : \mathcal{A}_\mu \rightarrow (-\mathcal{A}_0, \mathcal{A}_1, -\mathcal{A}_2), \quad T : \mathcal{A}_\mu \rightarrow (\mathcal{A}_0, -\mathcal{A}_1, -\mathcal{A}_2). \quad (4.7)$$

In addition to the U(1) gauge symmetry described above, one may introduce the usual electromagnetic interaction

$$L_{int}^{em} = q \left(\dot{x}_\mu A^\mu + \frac{1}{2} e i \xi_\mu \xi_\nu F^{\mu\nu} \right) \quad (4.8)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under P and T inversions the electromagnetic potential transforms as

$$P : A_\mu \rightarrow (A_0, -A_1, A_2), \quad T : A_\mu \rightarrow (-A_0, A_1, A_2).$$

It is the difference in the transformation law of the gauge potentials $\mathcal{A}_\mu(x)$ and $A_\mu(x)$ under the discrete P and T inversions what distinguishes the two types of U(1) gauge interactions.

At the Hamiltonian level the first class constraints (3.3), (3.4) are now modified as

$$\tilde{\phi} = \frac{1}{2} \left(\mathcal{P}^2 + m^2 - i \xi_\mu \xi_\nu (\mathcal{Q}\mathcal{N}\mathcal{F}^{\mu\nu} + qF^{\mu\nu}) \right) \approx 0, \quad (4.9)$$

$$\tilde{\chi} = i \epsilon_{\mu\nu\lambda} \mathcal{P}^\mu \xi^\nu \xi^\lambda - 2m\mathcal{N} \approx 0, \quad (4.10)$$

where

$$\mathcal{P}_\mu = p_\mu - \mathcal{Q}\mathcal{N}\mathcal{A}_\mu - qA_\mu.$$

These constraints satisfy the same algebra of the first class constraints of the free model, i.e. $\{\tilde{\phi}, \tilde{\chi}\} = 0$.

To conclude the analysis of the classical theory, let us comment on a property of the U(1) gauge symmetry (4.1), (4.3), (4.6). As it is shown in Appendix A, there is a hidden symmetry in the free theory ($\mathcal{A}_\mu = A_\mu = 0$), which is realized only in the sector of the variables θ_a . This symmetry leads to a ‘quantization’ condition on the parameter ν (here the maximal symmetry of the classical system is not required, thus ν can take any value). Namely, $\nu = \nu_{lk}$,

$$\nu_{lk} = \frac{2l+1}{2k+1}, \quad k, l \in Z. \quad (4.11)$$

This value of the parameter defines the ratio of the phase changes of the harmonic-like variables θ^\pm and $\xi^{(\pm)}$ for the complete time interval $\Delta\tau = \tau_f - \tau_i$ (see eq. (A.7)). As we have seen in section 3, the requirement of the maximal symmetry in the system selects from the infinite set (4.11) only the value $\nu = 1$. As a result, according to eq. (3.7), the phases of the harmonic-like variables $\xi^{(\pm)}$ and θ^\pm turn out to be locked (see eq. (3.22)); in fact, the ratio of the phase changes for the complete time interval should be equal to 1. After switching on the interaction with the U(1) gauge field \mathcal{A}_μ , the effect of the finite gauge transformation of

the variables θ^\pm , given by eq. (4.1) with the gauge function $\gamma(x)$ subject to the condition (4.6), only restores the initial freedom of the ratio of the phase changes of the harmonic-like variables. In fact, according to eqs. (A.5), (A.7), (4.1) and (4.6), for $\nu = 1$ one has

$$\theta^{\pm'}(\tau_f) = e^{\pm i(\Omega(\tau_f) + 2\pi n)} \theta^{\pm'}(\tau_i). \quad (4.12)$$

As a result, the ratio of the phase changes of the transformed variables $\theta^{\pm'}$ and of the variables $\xi^{(\pm)}$, given by (4.12), (A.5) and (A.7), is described again by (4.11) with $l = k + n$.

5 Quantization of the model

5.1 Free theory

Due to (3.1), the quantum operators associated with the odd variables must satisfy the following anticommutation relations:

$$[\hat{\xi}_\mu, \hat{\xi}_\nu]_+ = -\eta_{\mu\nu}, \quad [\hat{\theta}_a, \hat{\theta}_b]_+ = \delta_{ab}, \quad [\hat{\xi}_\mu, \hat{\theta}_a]_+ = 0.$$

Here and below we put $\hbar = 1$. The operators $\hat{\xi}^\mu$ and $\hat{\theta}_a$ are realized as

$$\hat{\xi}^\mu = \frac{1}{\sqrt{2}} \gamma^\mu \otimes \sigma_3, \quad \hat{\theta}_a = \frac{1}{\sqrt{2}} 1 \otimes \sigma_a, \quad a = 1, 2.$$

It is convenient to take the γ -matrices, satisfying the relation

$$\gamma_\mu \gamma_\nu = -\eta_{\mu\nu} + i\epsilon_{\mu\nu\lambda} \gamma^\lambda,$$

in the form $\gamma^0 = \sigma_3$, $\gamma^i = i\sigma^i$, $i = 1, 2$. The classical first class constraints (3.3), (3.4) turn into quantum equations singling out the physical subspace of the system:

$$\hat{\phi}\Psi = 0, \quad (5.1)$$

$$\hat{\chi}\Psi = 0. \quad (5.2)$$

In the construction of the quantum operator corresponding to the nilpotent constraint one has an ordering problem. For arbitrary ν , $\hat{\chi}$ is written as

$$\hat{\chi} = p\gamma + 2\nu m \hat{\mathcal{N}}. \quad (5.3)$$

If one starts with the classical expression $\mathcal{N} = \theta^+ \theta^-$, one has [11]:

$$\hat{\mathcal{N}} = \alpha \hat{\theta}^+ \hat{\theta}^- + (\alpha - 1) \hat{\theta}^- \hat{\theta}^+,$$

with α an arbitrary real parameter. For the sake of simplicity, in eq. (5.3) and below the operator of the energy-momentum vector of the system has been denoted in the same way as its classical counterpart. Upon multiplying the second quantum equation (5.2) with the operator $p\gamma - 2\nu m \hat{\mathcal{N}}$, one gets

$$(p^2 + 4\nu^2 m^2 \hat{\mathcal{N}}^2) \Psi = 0,$$

which reduces to the mass shell condition (5.1) (Klein-Gordon equation) iff

$$4\nu^2 \hat{\mathcal{N}}^2 = 1. \quad (5.4)$$

Eq. (5.4) is satisfied only when

$$\alpha = \frac{1}{2}, \quad \nu^2 = 1.$$

One immediately concludes that the choice (3.20) is a special one also in the quantum theory. The value $\alpha = 1/2$ leads to the operator

$$\hat{\mathcal{N}} = \frac{1}{2}[\hat{\theta}^+, \hat{\theta}^-] = -i\hat{\theta}_1\hat{\theta}_2 = \frac{1}{2} \cdot 1 \otimes \sigma_3. \quad (5.5)$$

Eq. (5.2) takes the form

$$(p\gamma \otimes 1 + m \cdot 1 \otimes \sigma_3)\Psi = 0. \quad (5.6)$$

Therefore, the components ψ_u and ψ_d of the function Ψ , $\Psi^t = (\psi_u^t, \psi_d^t)$, satisfy a pair of (2+1)-dimensional Dirac equations:

$$(p\gamma + m)\psi_u = 0, \quad (p\gamma - m)\psi_d = 0. \quad (5.7)$$

In correspondence with the definition of the (2+1)-dimensional spin operator,

$$\hat{S} = \frac{p\hat{J}}{\sqrt{-p^2}},$$

where \hat{J}_μ is the quantum operator corresponding to the total angular momentum of the system defined in eq. (3.8), one has

$$\hat{S} = -\frac{1}{2} \cdot \gamma^{(0)} \otimes 1. \quad (5.8)$$

Due to eqs. (5.7), the physical states ψ_u and ψ_d carry spin $+1/2$ and $-1/2$, respectively.

A P - and T -invariant quantum system emerges from the classical picture. In fact, starting from the classical relations (2.17)–(2.19), one finds the corresponding quantum transformation laws:

$$P : \Psi(x) \rightarrow \Psi'(x') = (\gamma^1 \otimes \sigma_1) \cdot \Psi(x), \quad x'_\mu = (x_0, -x_1, x_2), \quad (5.9)$$

$$T : \Psi(x) \rightarrow \Psi'(x') = (\gamma^0 \otimes \sigma_2) \cdot \Psi(x), \quad x'_\mu = (-x_0, x_1, x_2), \quad (5.10)$$

By direct inspection one easily verifies that eq. (5.6) is invariant with respect to these transformations.

Of course, the quantum system has also the charge conjugation symmetry

$$C : \Psi(x) \rightarrow \Psi_c(x) = (\sigma^1 \otimes 1) \cdot \Psi^*(x). \quad (5.11)$$

When the quantum system is interacting with the Abelian gauge fields $A_\mu(x)$ and $\mathcal{A}_\mu(x)$, under the charge conjugation symmetry one has to require that both such fields change sign.

Note that the system of equations (5.2) has also nontrivial solutions when $\nu \neq 1$ and α satisfy either the condition $(2\nu\alpha)^2 = 1$ or the condition $(2\nu(\alpha - 1))^2$. In these two cases one has only one nontrivial state given by the function ψ_u or ψ_d , respectively. The P - and T -invariance of the classical action is lost.⁴ For $\nu \neq 1$ one has an anomalous quantum scheme. For $\nu = 1$, one gets

$$\hat{\chi}^2 = -2\hat{\phi} + 4m\hat{\mathcal{N}}\hat{\chi}, \quad (5.12)$$

i.e. at the quantum level, unlike in the classical theory, the mass shell operator $\hat{\phi}$ and the operator $\hat{\chi}$ are dependent operators.

5.2 Super-extension of the Poincaré group

The quantum counterparts of the integrals (3.26) are:

$$\begin{aligned} \hat{R}_0 &= \frac{1}{4}(\gamma^{(0)} \otimes 1 - 1 \otimes \sigma_3), \\ \hat{R}_1 &= \frac{1}{4}(\gamma^{(2)} \otimes \sigma_1 - \gamma^{(1)} \otimes \sigma_2), \quad \hat{R}_2 = -\frac{1}{4}(\gamma^{(1)} \otimes \sigma_1 + \gamma^{(2)} \otimes \sigma_2). \end{aligned} \quad (5.13)$$

They satisfy the commutation relations of the $su(1, 1)$ algebra

$$[\hat{R}_\alpha, \hat{R}_\beta] = -i\epsilon_{\alpha\beta\gamma}\hat{R}^\gamma. \quad (5.14)$$

In the quantum theory

$$\hat{C} = \hat{R}_\alpha \hat{R}^\alpha = \frac{3}{8}(\gamma^{(0)} \otimes \sigma_3 - 1)$$

is the Casimir operator, which can be written as

$$\hat{C} = \frac{3}{4}((\hat{\mathcal{N}} - \hat{S})^2 - 1). \quad (5.15)$$

On the physical subspace it is reduced to $\hat{C} = -3/4$ (or $C = -\hbar^2 \cdot 3/4$ in units of \hbar). Hence, the operators \hat{R}_α , forming the $su(1, 1)$ algebra (5.14), act irreducibly on the physical subspace. The physical states ψ_u and ψ_d are the eigenstates of the operator

$$\hat{R}_0 = -\frac{1}{2}(\hat{\mathcal{N}} + \hat{S})$$

with the eigenvalues $-1/2$ and $+1/2$, respectively.

Since at the quantum level one has that:

$$\hat{\xi}^{(0)} \hat{R}_\alpha = -\frac{1}{\sqrt{2}} \hat{R}_\alpha, \quad (5.16)$$

the generators of the $SU(1, 1)$ symmetry, \hat{R}_α , coincide up to a c -number factor with the quantum operators corresponding to the $N = 3$ SUSY generators (3.33). Therefore, the operators \hat{R}_α satisfy not only the algebra (5.14), but also the $s(3)$ superalgebra [16]

$$[\hat{R}_\alpha, \hat{R}_\beta]_+ = \eta_{\alpha\beta} \cdot \frac{2}{3} \hat{C}, \quad [\hat{R}_\alpha, \hat{C}] = 0. \quad (5.17)$$

⁴ The discrete P - and T -symmetries were not discussed at the classical level in ref. [11] since it was assumed that the variables θ_a and v were scalars.

This corresponds to the classical algebra (4.7) of the SUSY generators (4.4), (4.6). Due to relation similar in form to (5.16), the quantum operator corresponding to the odd function ρ given by eq. (3.16) coincides (up to a c -number factor) with the quantum constraint $\hat{\chi}$.

Before going over to the analysis of the interacting quantum system, let us comment on the hidden global (super)symmetry. Upon writing down the quantum counterpart of the classical vector (3.32) as $\hat{\mathcal{R}}_\mu = e_\mu^{(0)} \hat{R}_0 + \hat{R}_\mu^\perp$, one sees that this operator satisfies the (anti)commutation relations

$$\hat{\mathcal{R}}_\mu \hat{\mathcal{R}}_\nu = \frac{1}{3} \eta_{\mu\nu} \hat{C} - \frac{i}{2} \epsilon_{\mu\nu\lambda} \hat{\mathcal{R}}^\lambda, \quad [\hat{\mathcal{R}}_\mu, \hat{C}] = 0. \quad (5.18)$$

Since $\hat{\mathcal{R}}_\mu$ is a Lorentz vector operator,

$$[\hat{J}_\mu, \hat{\mathcal{R}}_\nu] = -i \epsilon_{\mu\nu\lambda} \hat{\mathcal{R}}^\lambda,$$

and it commutes with the operator p_μ ,

$$[\hat{\mathcal{R}}_\mu, p_\nu] = 0,$$

the set of vector operators $\hat{\mathcal{R}}_\mu$ and p_μ supplied with the total angular momentum operator \hat{J}_μ form a super-extension of the Poincaré group, denoted as $\text{ISO}(2,1|3)$. The vector operator

$$\hat{\mathcal{J}}_\mu = \hat{J}_\mu - \hat{\mathcal{R}}_\mu$$

satisfies the same $su(1,1)$ algebra of the operators \hat{J}_μ and $\hat{\mathcal{R}}_\mu$; namely,

$$[\hat{\mathcal{J}}_\mu, \hat{\mathcal{J}}_\nu] = -i \epsilon_{\mu\nu\lambda} \hat{\mathcal{J}}^\lambda.$$

The Casimir operators of the supergroup $\text{ISO}(2,1|3)$ are the operators p^2 and $\hat{\mathcal{S}} = e_\mu^{(0)} \hat{\mathcal{J}}^\mu$. It is natural to name the operator

$$\hat{\mathcal{S}} = \frac{1}{2}(\hat{\mathcal{S}} - \hat{\mathcal{N}}) \quad (5.19)$$

the *superspin*. The operator $\hat{C} = \hat{\mathcal{R}}_\mu \hat{\mathcal{R}}^\mu$ can be written as a quadratic function of the superspin,

$$\hat{C} = 3\hat{\mathcal{S}}^2 - \frac{3}{4}.$$

Taking into account the explicit form of the operators $\hat{\mathcal{N}}$ and $\hat{\mathcal{S}}$ given by eqs. (5.5) and (5.8), one finds that the complete set of the eigenvalues of the superspin is given by the numbers $(-1/2, 0, 0, +1/2)$. The physical states ψ_u and ψ_d , carrying spin $+1/2$ and $-1/2$, are simultaneously the eigenstates of the superspin operator with zero eigenvalue, i.e. the one particle states of the quantum P, T -invariant theory realize an irreducible representation of the supergroup $\text{ISO}(2,1|3)$ labelled by the zero eigenvalue of the superspin.

The nonstandard character of the super-extension of the Poincaré group is related to the vector nature of the supercharge operators $\hat{\mathcal{R}}_\mu$, satisfying also the commutation relations of the $\text{SU}(1,1)$ generators.

In conclusion, we explicitly constructed the nontrivial ‘superposition’ of the discrete (P and T) and continuous ($U(1,1)$ and $S(3)$) (super)symmetries characteristic of the P, T -invariant planar fermion model. The generators of the continuous (super)symmetries are combined with the Poincaré generators resulting in the non-standard superextension of the Poincaré group.

5.3 Interacting quantum theory

In constructing the quantum operators corresponding to the first class constraints of the classical theory interacting with the U(1) gauge field \mathcal{A}_μ one should recall the form (5.5) of the operator $\hat{\mathcal{N}}$ since the physical states ψ_u and ψ_d , carrying opposite spins $+1/2$ and $-1/2$, are distinguished by this operator; in fact, these states carry opposite U(1) charges $+Q/2$ and $-Q/2$. Since, under P and T inversions, one has $\hat{\mathcal{N}} \rightarrow -\hat{\mathcal{N}}$, the physical states ψ_u and ψ_d change their U(1) charges under the discrete transformations. The difference between the U(1) gauge interaction we are considering here and the electromagnetic one lies in the fact that the states ψ_u and ψ_d , having one and the same electric charge q , are not distinguished by the electromagnetic interaction.

The quantum counterparts of the classical constraints (4.9) and (4.10) have the form

$$\hat{\chi}\Psi = 0, \quad \hat{\chi} = \hat{\mathcal{P}}\gamma \otimes 1 + m \cdot 1 \otimes \sigma_3, \quad (5.20)$$

$$\hat{\phi}\Psi = 0, \quad 2\hat{\phi} = \hat{\mathcal{P}}^2 + m^2 + \frac{1}{2}\epsilon_{\mu\nu\lambda}\gamma^\lambda \otimes 1 \cdot \left(\frac{1}{2}\mathcal{Q}\mathcal{F}^{\mu\nu} \cdot 1 \otimes \sigma_3 + qF^{\mu\nu} \cdot 1 \otimes 1 \right), \quad (5.21)$$

where $\hat{\mathcal{P}}_\mu = p_\mu - \mathcal{Q}\hat{\mathcal{N}}\mathcal{A}_\mu(x) - qA_\mu(x)$. The operators $\hat{\chi}$ and $\hat{\phi}$ satisfy the initial trivial algebra, i.e. $[\hat{\phi}, \hat{\chi}] = 0$, and the equation (5.12) is still valid when the fermions are interacting with the chiral and electromagnetic U(1) fields.

The field Lagrangian describing the interacting theory is

$$\mathcal{L}(x) = \bar{\Psi}(x)\hat{\chi}\Psi(x), \quad (5.22)$$

where $\bar{\Psi} = \Psi^\dagger \cdot \gamma^0 \otimes 1$, and $p_\mu = -i\partial_\mu$. The equations of motion derived from the Lagrangian (5.22) are given by eq. (5.20), whereas the eq. (5.21), quadratic in ∂_μ , comes as a consequence of equation (5.12). The field Lagrangian (5.22) has been used in some models of high- T_c superconductivity [2].

6 Concluding remarks

In this paper we showed how the hidden U(1,1) symmetry and the hidden $N = 3$ supersymmetry of the P, T -invariant quantum fermion model described by (1.1) could be understood in terms of the 3D pseudoclassical model described by eqs. (2.1)–(2.5). In particular, our approach clarified the nature of the pseudounitary U(1,1) symmetry appearing in the quantum model since — at the classical level — the two pairs of oscillator-like odd variables θ^\pm and $\xi^{(\pm)}$, from which the generators of the U(1,1) symmetry are constructed, have Poisson brackets of opposite sign. Furthermore, the pseudoclassical model clearly shows that the generators of the SU(1,1) symmetry and the odd generators of the $N = 3$ supersymmetry are intimately related; in fact, they only differ in the factor $\xi^{(0)}$ which is an integral of motion. For the quantum model, the generators of the SU(1,1) symmetry and those of the $N = 3$ supersymmetry coincide up to a c -number factor since they are eigenvectors of the operator $\hat{\xi}^{(0)}$ corresponding to the same eigenvalue.

The continuous symmetries of the pseudoclassical and P, T -invariant quantum fermion model are identical if one requires that the classical model has a maximal symmetry; it is

this requirement the one selecting the special value ($\nu = 1$) of the parameter appearing in the pseudoclassical model. It would be interesting to see how a special value of the parameter ν may be selected in the path-integral approach. Unfortunately, this is a difficult task [17] since the model has an even nilpotent constraint for which a gauge fixing condition cannot be introduced.

Interesting features appear also when one supplement the pseudoclassical model with a local $U(1)$ gauge symmetry which is the classical analog of the corresponding quantum $U_c(1)$ chiral gauge symmetry. As it is shown in Appendix A, when the parameter ν is not restricted by the condition $\nu = 1$, in the ungauged model there is a hidden symmetry realized in the sector of the θ^\pm variables, which is responsible for a sort of quantization condition on the parameter ν :

$$\nu = \nu_{lk} = \frac{2l+1}{2k+1}, \quad l, k \in Z.$$

This symmetry is effectively “restored” in the gauge model even when the maximal continuous symmetry of the free pseudoclassical model ($\nu = 1$) is required. It seems to be interesting to look for the quantum analog of this hidden symmetry and reveal its role in the context of the field model with the gauged $U_c(1)$ chiral symmetry.

The quantum generators of the $N=3$ supersymmetry together with the energy-momentum vector and the total angular momentum vector operators are the generators of a superextension of the $(2+1)$ -dimensional Poincaré group. The complete set of possible eigenvalues of the superspin operator, which is one of the Casimir operators of this supergroup, is given by the values $(-1/2, 0, 0, +1/2)$. The physical subspace of the quantum P, T -invariant fermion model realizes an irreducible representation of the Poincaré supergroup, labelled by the zero eigenvalue of the superspin. In this representation the states of the P, T -invariant system carry spin $-1/2$ and $+1/2$.

In the analysis of the $U(1,1)$ symmetry given in section 3, either the variables θ_a or the variables $\xi^{(i)} = \xi^\mu e_\mu^{(i)}$ are not $U(1,1)$ covariant quantities; only their specific linear combinations Σ_a^i , defined in eq. (3.36), are covariant under the action of $U(1,1)$. One may then expect that the model (2.1)–(2.5), written in terms of the latter set of variables, is manifestly $U(1,1)$ covariant and shows explicitly the hidden $N=3$ supersymmetry. However, since $\{x_\mu, \Sigma_a^i\} \neq 0$, the reformulation of the system in a manifestly $U(1,1)$ covariant way is a nontrivial task within the Lagrangian formalism.

In conclusion, the simple pseudoclassical model considered in this paper reveals a non-trivial ‘superposition’ of discrete and continuous, global and local, ordinary and graded symmetries, and allows for a natural way to construct a field Lagrangian with a $U(1)$ gauge field coupled to the chiral current. One of our motivation to investigate the properties of this particular pseudoclassical model lies in the fact that the planar SPM [7] does not naturally reproduce at the classical level the symmetries of the P, T -invariant 3D quantum fermion model of ref. [4]. Moreover, in the SPM there is no natural analog of the $U(1)$ chiral symmetry we explicitly constructed, since this symmetry acts only in the subspace of the θ^\pm variables. Due to this, a gauge generalization based on the SPM is not immediate. Thus, the pseudoclassical model of ref. [11] provides the simplest classical example of a model exhibiting the symmetries of the P, T -invariant planar quantum free fermion model.

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A Hidden symmetry

There is a ‘hidden’ symmetry in the free model (2.1), induced by transformations in the subspace of the variables θ^\pm . It is a product of one of the discrete transformations defined by eq. (2.19), say the P inversion, and the special $U(1)$ transformation

$$U_s : \theta^\pm \rightarrow \theta^{\pm'} = e^{\pm i\gamma_s(\tau)} \theta^\pm, \quad (\text{A.1})$$

with $\gamma_s(\tau)$ given by:

$$\gamma_s(\tau) = -2\nu\Omega(\tau), \quad \Omega(\tau) = 2m \int_{\tau_i}^{\tau} v(\tau') d\tau'. \quad (\text{A.2})$$

In (A.2) it is assumed that $\nu \neq 0$. Under the transformation $U_s P_\theta$, — P_θ being a restriction of the P inversion to the sector of the variables θ^\pm , i.e. $P_\theta : \theta_a \rightarrow \theta'_a = (\theta_1, -\theta_2)$ — the Lagrangian is invariant: $U_s P_\theta : L \rightarrow L$. The boundary term B is invariant under this transformation $U_s P_\theta$ iff the function $\gamma_s(\tau)$ is such that

$$\Delta\gamma_s = \gamma_s(\tau_f) - \gamma_s(\tau_i) = \gamma(\tau_f) = 2\pi n, \quad n \in Z, \quad (\text{A.3})$$

which amounts to

$$\nu\Omega(\tau_f) = \pi n, \quad n \in Z. \quad (\text{A.4})$$

This condition is not affected by the local transformations (2.14) and (2.15) due to the boundary conditions (2.16) imposed on the parameters α and β .

In accordance with eqs. (3.7), (3.19), one gets

$$\xi^{(\pm)}(\tau_f) = \xi^{(\pm)}(\tau_i) \cdot e^{\pm i\Omega(\tau_f)}, \quad \theta^\pm(\tau_f) = \theta^\pm(\tau_i) \cdot e^{\pm i\nu\Omega(\tau_f)}. \quad (\text{A.5})$$

The boundary conditions (2.7) require that the quantities

$$\theta_{1/2}^\pm = \theta^\pm(\tau_i) + \theta^\pm(\tau_f), \quad \xi_{1/2}^{(\pm)} = \xi^{(\pm)}(\tau_i) + \xi^{(\pm)}(\tau_f)$$

must be fixed. In terms of the above variables, one has that

$$\xi^{(\pm)}(\tau_i) = \left(1 + e^{\pm i\Omega(\tau_f)}\right)^{-1} \cdot \xi_{1/2}^{(\pm)}, \quad \xi^{(\pm)}(\tau_f) = \left(1 + e^{\mp i\Omega(\tau_f)}\right)^{-1} \cdot \xi_{1/2}^{(\pm)}, \quad (\text{A.6})$$

and similar equations for θ^\pm with the corresponding change of $\Omega(\tau_f)$ in $\nu\Omega(\tau_f)$. When $\xi_{1/2}^{(\pm)} \neq 0$ or (and) $\theta_{1/2}^\pm \neq 0$, and one (or both) of the equations

$$\Omega(\tau_f) = \pi(2k+1), \quad \nu\Omega(\tau_f) = \pi(2l+1), \quad k, l \in Z, \quad (\text{A.7})$$

hold, (A.6) cannot be satisfied since the denominators vanish. Thus, either one prohibits $\Omega(\tau_f)$ to obey eqs. (A.7), or one requires that

$$\theta_{1/2}^\pm = 0, \quad \xi_{1/2}^{(\pm)} = 0. \quad (\text{A.8})$$

If (A.8) is satisfied, the conditions (A.7) must be also satisfied. We choose the second possibility, i.e. we assume antiperiodic boundary (in evolution parameter) conditions for the Grassmann ‘oscillating’ variables $\theta^\pm(\tau)$ and $\xi^{(\pm)}(\tau)$. The conditions (A.7) lead to a sort of quantization condition $\nu = \nu_{lk}$,

$$\nu_{lk} = \frac{2l+1}{2k+1}, \quad k, l \in Z, \quad (\text{A.9})$$

which includes the special case $\nu^2 = 1$. For the set of values of the parameter ν given by eq. (A.9), the boundary conditions (A.4) are satisfied, and, hence, the hidden symmetry described above is realized.

For $\nu = 1$, the model described by the Lagrangian $\tilde{L} = L + L_{int} + L_{int}^{em}$, where L , L_{int} and L_{int}^{em} are given by eqs. (2.1), (4.5) and (4.8), is invariant under the hidden symmetry $U_s P_\theta$, if one supplements the transformation properties of the pseudoscalar variables with the transformation law $U_s P_\theta : (\mathcal{A}_\mu, A_\mu) \rightarrow (-\mathcal{A}_\mu, A_\mu)$ for the U(1) gauge fields.

B Standard 3D pseudoclassical model

Here we shall show that, though the quantization of the standard pseudoclassical model for the massive relativistic spin particle [7] also leads to the P, T -invariant 3D fermion quantum model, nevertheless, the SPM does not reproduce either the U(1,1) symmetry or the $N = 3$ supersymmetry at the classical level.

The Lagrangian of the model [7] is similar in form to the one of the model we have considered in this paper, namely:

$$L = \frac{1}{2e}(\dot{x}_\mu - i\lambda\xi_\mu)^2 - \frac{e}{2}m^2 - im\lambda\xi_* - \frac{i}{2}\xi_\mu\dot{\xi}^\mu - \frac{i}{2}\xi_*\dot{\xi}_*.$$

There is an odd Lagrange multiplier λ instead of the even multiplier v , and one odd (pseudo)scalar variable ξ_* instead of the pair of variables θ_a , $a = 1, 2$. The total Hamiltonian of the system is given by:

$$H = e\phi + i\lambda\chi + up_e + \rho\pi_\lambda.$$

It is a linear combination of the primary constraints $p_e \approx 0$ and $\pi_\lambda \approx 0$, π_λ being the canonical momentum conjugate to λ , of the constraint ϕ given by eq. (3.3) and of

$$\chi = p\xi + m\xi_* \approx 0. \quad (\text{B.1})$$

The brackets of ξ_μ are given by the same eq. (3.1), whereas

$$\{\xi_*, \xi_*\} = i.$$

The equations of motion for the odd variables are $\dot{\xi}_\mu = \lambda p_\mu$ and $\dot{\xi}_* = m\lambda$. The quantities $\xi^{(i)}$, $i = 1, 2$, and $\xi^{(0)}\xi_*$ are integrals of motion. The last even integral is weakly equal to zero if one takes into account the constraint (B.1), but it is a nontrivial operator at the quantum level.

Let us briefly consider the quantum theory of this model, and then return to the classical model in order to construct the classical counterparts of the quantum symmetry generators. The quantum operators corresponding to the variables ξ_μ and ξ_* may be realized as

$$\hat{\xi}_\mu = \frac{1}{\sqrt{2}}\gamma_\mu \otimes \sigma_1, \quad \hat{\xi}_* = -\frac{i}{\sqrt{2}}1 \otimes \sigma_2. \quad (\text{B.2})$$

The internal scalar product is defined as $(\Psi_1, \Psi_2) = \bar{\Psi}_1 \Psi_2$, with $\bar{\Psi} = \sqrt{2}\Psi^+ \hat{\xi}^0$. With this indefinite scalar product all the operators (B.2) are hermitian. The quantum counterpart of the constraint (B.1) is

$$\frac{1}{\sqrt{2}}1 \otimes \sigma_1(p\gamma \otimes 1 + m1 \otimes \sigma_3)\Psi = 0, \quad (\text{B.3})$$

which is equivalent to eq. (5.6). If one takes $\mathcal{L} = \sqrt{2}\bar{\Psi}\hat{\chi}\Psi$ as the field Lagrangian, one finds that it coincides with the Lagrangian (5.22) (with $A_\mu = \mathcal{A}_\mu = 0$). Hence, the quantization of the SPM reproduces the P, T -invariant 3D fermion system.

The classical quantities corresponding to the generators (5.13) are given by :

$$R_0 = \frac{i}{2}\xi^{(1)}\xi^{(2)} \cdot G, \quad R_1 = \frac{1}{2\sqrt{2}}\xi^{(2)} \cdot G, \quad R_2 = -\frac{1}{2\sqrt{2}}\xi^{(1)} \cdot G, \quad (\text{B.4})$$

where $G = 1 - 2\xi^{(0)}\xi_*$. First of all, one sees that these quantities have different Grassmann parity: R_0 is even, whereas R_i , $i = 1, 2$, are odd; in addition, one can check that they do not form an algebra with respect to the brackets. The classical counterpart of the operator $\gamma^{(0)} \otimes \sigma_3$ is, up to a c-number factor, the even quantity $\xi^{(0)}\xi_*$; therefore, one has no possibility to ‘change’ the Grassmann parity of the integrals of motion as it was possible for the model (2.1). Due to this, in this model one cannot reproduce the $N = 3$ supersymmetry at the classical level.

One may start not from the quantum constraint given by eq. (5.6) and equivalent to equation (B.3), but from the quantum constraint (B.3) itself, which is equal to the constraint (5.6) multiplied by $1 \otimes \sigma_1$. One can then take the corresponding operator $\hat{\mathcal{X}} = \gamma^{(0)} \otimes \sigma_1 - i1 \otimes \sigma_2$ as the generator of the $U(1)$ symmetry. In this case one has the following set of generators of the $SU(1,1)$ symmetry, commuting with the $U(1)$ generator:

$$\Gamma^{(0)} = -\frac{1}{2}\gamma^{(0)} \otimes 1, \quad \Gamma^{(i)} = -\frac{1}{2}\gamma^{(i)} \otimes \sigma_2. \quad (\text{B.5})$$

From these one constructs the Lorentz vector $\Gamma_\mu = e_\mu^{(\alpha)}\Gamma_{(\alpha)}$ satisfying the equation

$$\Gamma_\mu \Gamma_\nu = -\frac{1}{4}\eta_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\lambda}\Gamma^\lambda.$$

The anticommutator of these operators is a constant; the nontrivial supersymmetry of the generators is lost. Due to this fact, instead of (5.19), in the SPM one has $\hat{\mathcal{S}} = \hat{\mathcal{J}}^{(0)} = 0$,

$\hat{\mathcal{J}}_\mu = \hat{J}_\mu - \Gamma_\mu$. Again, the classical analogs of the SU(1,1) generators (B.5), given by $\Gamma^{(0)} = -i\xi^{(1)}\xi^{(2)}$ and $\Gamma^{(i)} = -\sqrt{2}\epsilon^{ij}\xi^{(0)}\xi^{(j)}\xi_*$, do not reproduce the SU(1,1) algebra at the classical level.

In the SPM there is no natural analog of the U(1) symmetry acting in the subspace of the variables θ^\pm . Furthermore, using the SPM, it is cumbersome to provide a natural construction of the U(1) gauge theory. Thus, the SPM does not reproduce the symmetries of the quantum P, T -invariant 3D fermion theory at the classical level.

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